Propagation of Thermoelastic Disturbances in Non-Fourier Solids

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Integral transform techniques are employed to study the influence of small thermoelastic coupling on the propagation of thermoelastic disturbances in an infinite medium. The thermal relaxation time of heat conduction is also included in the analysis to insure that the thermal waves propagate with finite signal speeds. Explicit expressions for the wave speeds and amplitudes are obtained. Numerical results are presented to illustrate the salient features of the problem.

Introduction

IT is now known that in some materials at low temperatures thermal disturbances propagate with finite speeds. Extensive reviews of the existing experimental investigations and verifications may be found in many articles of which we mention those of Nayfeh and Nemat-Nasser[1,2] and the more recent article by Atkin et al.[3] However, until quite recently, fluid helium was the only substance known to exhibit a wave-like heat flow (second sound) at temperatures below 2.2 K (see Ref. 4). In an attempt to explain this phenomenon, Landau[5] suggested that the behavior of superfluid helium may be described by a gas of elementary excitations called phonons, and that a thermal wave is the propagation of a phonon-density disturbance. Landau's theory predicts that this second sound propagates with the speed $v_p/\sqrt{3}$, where v_p is the velocity of ordinary sound (first sound). This prediction has been verified both theoretically[6,7,8] and experimentally.[9] It was predicted then that second sound must exist in any solid since all solids exhibit phonon excitations (see Ref. 10). However, since most solids have chemical and isotropic impurities and mechanical defects, the phonons are scattered strongly by these impurities resulting in a diffusion-like process of heat flow. The reason for the observation of second sound in solids such as helium and sodium fluoride is that these substances can be reduced to an extremely pure form at low temperatures.

In a number of efforts a phenomenologic modification is introduced in the classical Fourier law of heat conduction, and a wave-type equation for heat transport is obtained (see for example Ref. 3 for the latest list of relevant references).

Based on the pheonomenologic approach, several aspects of thermoelastic waves in solids have been considered by a number of authors. Nayfeh and Nemat-Nasser[1] considered the propagation of plane thermoelastic waves in unbounded media as well as Rayleigh's surface waves along a half-space. As an application of the results in Ref. 1, the same authors extended the analysis to study the transient behavior of thermoelastic waves due to an instantaneous heat source applied to the free surface of a half-space (a two-dimensional generalized Lamb problem).[2] Lord and Shulman[11] studied the transient behavior of the one-dimensional thermoelastic waves. Norwood and Warren[12] pointed out certain inconsistencies in Ref. 11 and proceeded to study wave-front and long time approximations for a half-space subjected to various boundary conditions.

In this paper, we study the distribution of the stress and temperature fields due to the application of an instaneous heat source in an unbounded medium. Since for most conceivable situations the thermoelastic coupling coefficient is small, we express our results in terms of this coefficient and obtain explicit solutions for all ranges of time.† We also present numerical results for the stress and temperature to illustrate the salient features of this problem.

Formulation of Problem and Basic Relations

Consider an infinite thermally conducting elastic medium that is initially at rest and has a uniform temperature. Our objective is to determine the stress and the temperature distributions when this solid is subjected to an instantaneous heat source distributed on a plane. Without loss of generality, we choose this plane to be perpendicular to the x direction. Hence, from symmetry, we consider the one-dimensional case, in the x direction, where all physical quantities are functions of x and t only. With this notation, we summarize the one-dimensional thermoelastic field equations with thermal relaxation which are needed to study the present problem as:

$$\rho \ddot{u} = (\lambda + 2\mu) u'' - \gamma \theta' \tag{1}$$

Energy Equation

$$\rho c_v (\dot{\theta} + \tau_0 \ddot{\theta}) + \gamma \theta_0 \; (\dot{u} + \tau_0 \ddot{u}') - \kappa \theta'' = q_0 \delta(x) \delta(t)$$

Constitutive Relation

$$T = (\lambda + 2\mu) u' - \gamma \theta \tag{3}$$

where ρ is the mass-density, u is the displacement, λ and μ are the Lamé elastic constants; $\gamma = \alpha(3\lambda + 2\mu)$, where α is the coefficient of linear thermal expansion; θ denotes the change in the absolute basic temperature θ_0; superposed primes and dots stand for differentiation with respect to x and t, respectively; c_v is the specific heat at constant volume; τ_0 is the thermal relaxation time for the acceleration of heat flow; κ is the coefficient of thermal conductivity; q_0 is a constant; $\delta(-)$ is the Diract delta; and T is the normal stress.‡ For a detailed development of the general three-dimensional field equations of thermoelasticity that also include the thermal relaxation time, we refer the reader to Ref. 1. When τ_0

†It has come to our attention that Kao[13] has recently treated the uncoupled case where the thermoelastic coupling coefficient was neglected.

‡Here the function $q_0 \delta(x) \delta(t)$ corresponds to the Green's function of the general forcing function $[Q(t) + t_0 \dot{Q}(t)] \; \delta(x)$ which would appear in Eq. (2) for a heat source of intensity $Q(t)\delta(x)$. Solution for the general source can of course be obtained with the proper Laplace transform convolution and the Green function's results.

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vanishes, the present theory reduces to the classical theory of thermoelasticity treated by Boley and Tolins.[14]

For the convenience of analysis, we introduce the following notation

$$\omega^* = \frac{\rho c_v v_p^2}{\kappa} \qquad v_p^2 = \frac{\lambda+2\mu}{\rho} \qquad \epsilon = \frac{\gamma^2\theta_0}{\beta^2\mu\rho c_v}$$

$$\beta^2 = \frac{\lambda+2\mu}{\mu} \qquad \tau=\tau_0\omega^*$$

and use

$$\frac{1}{\omega^*} \qquad \frac{v_p}{\omega^*} \qquad \frac{\rho c_v v_p}{\gamma\omega^*} \qquad \frac{\rho c_v(\lambda+2\mu)}{\gamma}$$

and θ_0 as the units of time, length, displacement, stress, and temperature, respectively. We then arrive at the following set of dimensionless field equations:

$$\dot{\theta}+\tau\ddot{\theta}-\theta''+\dot{u}'+\tau\ddot{u}'=q\delta(x)\delta(t) \tag{4}$$

$$u''-\ddot{u}-\epsilon\theta'=0 \tag{5}$$

$$T-u'+\epsilon\theta=0 \tag{6}$$

where ϵ is the thermoelastic coupling coefficient, τ is the thermal relaxation parameter,§ $q=q_0v_o^2/\kappa\theta_0\omega^{*2}$, and where the same letter is used to denote the corresponding dimensionless quantity.

Solutions

To determine the solutions of this problem, we apply to the system of Eqs. (4-6) the one-sided Laplace transform with respect to time, and the exponential Fourier transform with respect to the x coordinate. The appropriate solutions of the resulting equations are to be constructed and then inverted.

The one-sided Laplace transform and the exponential Fourier transform are defined, respectively, as

$$\mathcal{L}[\phi(x,t)]=\int_0^\infty \phi(x,t)e^{-pt}\mathrm{d}t=\tilde{\phi}(x,p) \tag{7}$$

$$\mathfrak{F}[\tilde{\phi}(x,p)]=\int_{-\infty}^\infty \tilde{\phi}(x,p)e^{-i\xi x}\mathrm{d}x=\hat{\phi}(\xi,p) \tag{8}$$

Application of Eqs. (7) and (8) to Eqs. (4-6) yields

$$\hat{\theta}=\frac{q(p^2+\xi^2)}{(\xi^2+\alpha_1^2)(\xi^2+\alpha_2^2)} \tag{9}$$

$$\hat{u}=\frac{-\epsilon i\xi q}{(\xi^2+\alpha_1^2)(\xi^2+\alpha_2^2)} \tag{10}$$

$$\hat{T}=\frac{-\epsilon qp^2}{(\xi^2+\alpha_1^2)(\xi^2+\alpha_2^2)} \tag{11}$$

where¶

$$\alpha_1^2+\alpha_2^2=p^2[1+\tau^*(1+\epsilon)] \tag{12a}$$

$$\alpha_1^2\alpha_2^2=p^4\tau^* \qquad \tau^*=\tau+(1/p) \tag{12b,c}$$

§For some physical interpretations of the parameter τ, see Refs. 1 and 2.

¶In taking the Fourier transform, it is assumed that all field quantities are bounded as $x\to\infty$.

The relations (12) are equivalent to Eq. (12) in Ref. 1. Equations (9) and (11) may be written as

$$\hat{\theta}=\frac{q}{(\alpha_1^2-\alpha_2^2)}\left[\frac{(p^2-\alpha_2^2)}{(\xi^2+\alpha_2^2)}-\frac{(p^2-\alpha_1^2)}{(\xi^2+\alpha_1^2)}\right] \tag{13}$$

$$\hat{T}=\frac{\epsilon qp^2}{(\alpha_1^2-\alpha_2^2)}\left[\frac{1}{(\xi^2+\alpha_1^2)}-\frac{1}{(\xi^2+\alpha_2^2)}\right] \tag{14}$$

The inverse Fourier transforms of Eqs. (13) and (14) are readily obtained. We have

$$\tilde{\theta}=\frac{q}{2(\alpha_1^2-\alpha_2^2)}\left[\frac{p^2-\alpha_2^2}{\alpha_2}e^{-\alpha_2x}-\frac{(p^2-\alpha_1^2)}{\alpha_1}e^{-\alpha_1x}\right] \tag{15}$$

$$\tilde{T}=\frac{\epsilon qp^2}{2(\alpha_1^2-\alpha_2^2)}\left[\frac{e^{-\alpha_1x}}{\alpha_1}-\frac{e^{-\alpha_2x}}{\alpha_2}\right] \tag{16}$$

which, with the help of Eq. (12b), further reduce to

$$\tilde{\theta}=\frac{q}{2p^2\sqrt{\tau^*}(\alpha_1^2-\alpha_2^2)}\times[\alpha_1(p^2-\alpha_2^2)e^{-\alpha_2x}-\alpha_2(p^2-\alpha_1^2)e^{-\alpha_1x}] \tag{17}$$

$$\tilde{T}=\frac{\epsilon q}{2\sqrt{\tau^*}(\alpha_1^2-\alpha_2^2)}[\alpha_2e^{-\alpha_1x}-\alpha_1e^{-\alpha_2x}] \tag{18}$$

In order to find the inverse Laplace transform of Eqs. (17) and (18), explicit expressions for α_1 and α_2 must be found. This, however, is not possible, and only approximate values can be found. Since for most conceivable situations the thermoelastic coupling coefficient ϵ is small (in particular the value $\epsilon=0.03$ used in Ref. 12, we express our results in terms of this coefficient. To this end, we observe that Eqs. (12a,b) pertain to the coupled dilatational and thermal waves. To find explicit solutions for α_1 and α_2, we seek solutions of Eqs. (12a,b) for small values of ϵ. A straightforward expansion in powers of ϵ yields

$$\alpha_1^2=\tau^*p^2\left[1+\frac{\tau^*}{\tau^*-1}\epsilon-\frac{\tau^{*2}}{(\tau^*-1)^3}\epsilon^2+\ldots\right] \tag{19}$$

$$\alpha_2^2=p^2\left[1-\frac{\tau^*}{\tau^*-1}\epsilon+\frac{\tau^{*3}}{(\tau^*-1)^3}\epsilon+\ldots\right] \tag{20}$$

The solutions (19,20) constitute good approximations to the exact results as long as $(\tau^*-1)^2\neq 0(\epsilon)$, or equivalently $\tau-1+(1/p)\neq 0(\epsilon^{1/2})$. Since $\tau\to 0$, this condition is violated only when $(\tau-1)^2=0(\epsilon)$ with $p\to\infty$. Most existing evidence suggests that the value of τ is not close to 1 and expressions (19,20) constitute good approximations to the exact solutions and thus will be employed exclusively in our subsequent considerations. Taking the square roots of Eqs. (10) and (20), using Eq. (12c), and employing the positive sign (to insure boundedness at infinity), to the first-order of approximation in ϵ, we finally arrive at

$$\alpha_1=\sqrt{\tau p^2+p}\left[1+\frac{(\tau p+1)\epsilon}{2\{1+p(\tau-1)\}}\right] \tag{21}$$

$$\alpha_2=p\left[1-\frac{(\tau p+1)\epsilon}{2\{1+p(\tau-1)\}}\right] \tag{22}$$

On substitution from Eqs. (21) and (22) into Eqs. (17) and (18), and after some rather lengthy algebraic reductions to the first-order of approximation in ϵ, we obtain

$$\tilde{\theta}=\frac{q}{2\sqrt{\tau}}\left[\left\{1-\epsilon\frac{(\tau+1)\tau}{2(\tau-1)^2}\right\}+\epsilon\frac{(3\tau+1)}{2(\tau-1)^3\left\{p+\frac{1}{\tau-1}\right\}}-\frac{\epsilon}{(\tau-1)^4\left\{p+\frac{1}{\tau-1}\right\}^2}\right]\frac{\exp\left(-\sqrt{p\left(p+\frac{1}{\tau}\right)}\sqrt{\tau}x\right)}{\sqrt{p\left(p+\frac{1}{\tau}\right)}}$$

$$+\frac{q\epsilon x}{4(1-\tau)}\left[\tau+\frac{1}{(1-\tau)\left\{p+\frac{1}{\tau-1}\right\}}\right]\exp\left(-\sqrt{p\left(p+\frac{1}{\tau}\right)}\sqrt{\tau}x\right)+\frac{q\epsilon}{2(\tau-1)^2}\left[\frac{\tau}{p+\frac{1}{\tau-1}}-\frac{1}{(\tau-1)\left\{p+\frac{1}{\tau-1}\right\}^2}\right]\exp(-px)\cdot \qquad (23)$$

$$\bar{T}=\frac{q\epsilon}{2(1-\tau)\left\{p+\frac{1}{\tau-1}\right\}}\exp(-px)+\frac{q\epsilon}{2\sqrt{\tau}(\tau-1)}\left[1+\frac{1}{(1-\tau)\left\{p+\frac{1}{\tau-1}\right\}}\right]\frac{\exp\left(-\sqrt{p\left(p+\frac{1}{\tau}\right)}\sqrt{\tau}x\right)}{\sqrt{p\left(p+\frac{1}{\tau}\right)}} \qquad (24)$$

With the help of the shifting and convolution formulas, together with tabulated formulas (see Ref. 15, formula 2.4.183) we obtain the inverse Laplace transforms of Eqs. (23) and (24) as

$$\theta=\frac{q}{2\sqrt{\tau}}\exp-\left(\frac{t}{2\tau}\right)I_0(\eta)H(t-\sqrt{\tau}x)+\frac{q\epsilon}{2(\tau-1)^2}\left[\left(\tau-\frac{t-x}{\tau-1}\right)\exp\left(\frac{t-x}{1-\tau}\right)H(t-x)-\frac{\sqrt{\tau}}{2}\exp\left(-\frac{t}{2\tau}\right)\left\{(\tau+1)I_0(\eta)\right.\right.$$

$$\left.\left.+\frac{(1-\tau)}{4}x^2\frac{I_1(\eta)}{\eta}\right\}H(t-\sqrt{\tau}x)+\exp\left(\frac{t}{1-\tau}\right)\left\{\int_{\sqrt{\tau}x}^{t}\exp\left[\frac{u(\tau+1)}{2\tau(\tau-1)}\right]\left[\frac{4}{\tau-1}\left(\frac{3\tau+1}{2}-\frac{t-u}{\tau-1}\right)I_0\xi(u)-\frac{x^2}{2\tau}I_1\frac{(\xi(u))}{\xi}\right]\mathrm{d}u\right\}\right] \qquad (25)$$

$$T=\frac{\epsilon q}{2(1-\tau)}\left[\exp\left(\frac{t-x}{1-\tau}\right)H(t-x)-\frac{1}{\sqrt{\tau}}\right.$$

$$\times\exp\left(-\frac{t}{2\tau}\right)I_0(\eta)H(t-\sqrt{\tau}x)-\frac{1}{\sqrt{\tau}(1-\tau)}\exp\left(\frac{t}{1-\tau}\right)$$

$$\left.\times\int_{\sqrt{\tau}x}^{t}\exp\left[\frac{u(\tau+1)}{2\tau(\tau-1)}\right]I_0(\xi(u))\mathrm{d}u\right] \qquad (26)$$

where

$$\xi(u)=\frac{(u^2-\tau x^2)^{1/2}}{2\tau} \qquad \eta=\frac{(t^2-\tau x^2)^{1/2}}{2\tau}$$

In inverting Eqs. (23) and (24), we have used the key formula

$$\mathcal{L}^{-1}\left[\frac{\exp\left(-x\sqrt{(p+2\alpha)(p+2\beta)}\right)}{\sqrt{(p+2\alpha)(p+2\beta)}}\right]$$

$$=\exp-\rho t/2\; I_0\left(\frac{\sigma}{2}\sqrt{t^2-x^2}\right)H(t-x)$$

where $\rho=\alpha+\beta$, $\sigma=\alpha-\beta$, $H(t-x)$ is the Heaviside step function, I_0 is the Bessel function of the zeroth kind; and the fact

$$\mathcal{L}^{-1}\exp\left[-x\sqrt{(p+\alpha)(p+\beta)}\right]=$$

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathcal{L}^{-1}\frac{\exp\left(-x\sqrt{(p+\alpha)(p+\beta)}\right)}{\sqrt{(p+\alpha)(p+\beta)}}\right]$$

Inspection of solutions (25) and (26) shows that they consist of two waves propagating in the x direction at different velocities. The thermal wave-front propagates with the velocity equal to $1/\sqrt{\tau}$ and the stress wave-front propagates with velocity equal to unity. The thermal wave travels faster or slower than the stress wave depending upon whether $\tau<1$ or $\tau>1$.

For the special case $\epsilon=0$, the stress vanishes and the temperature reduces to

$$\theta=\frac{q}{2\sqrt{\tau}}\exp(-t/2\tau)\; I_0\left(\frac{1}{2\tau}\sqrt{t^2-x^2}\right)H(t-\sqrt{\tau}x) \qquad (27)$$

This result corresponds to the Green's function of the general solution obtained by Mauer and Thompson.[16] In fact, for their choice $\phi(t)=q_0\delta(t)$, our result (27) would have to be convoluted with $(1+\tau p)$, resulting in the identical results reported in Ref. 16. The result (27) resembles the solution obtained by Lee and Canter[17] for stress wave propagation in a viscoelastic medium. For a fixed τ, Eq. (27) shows that the temperature is discontinuous at its wave front. The magnitude of this discontinuity can be obtained by setting $t=\sqrt{\tau}\,x$ in Eq.

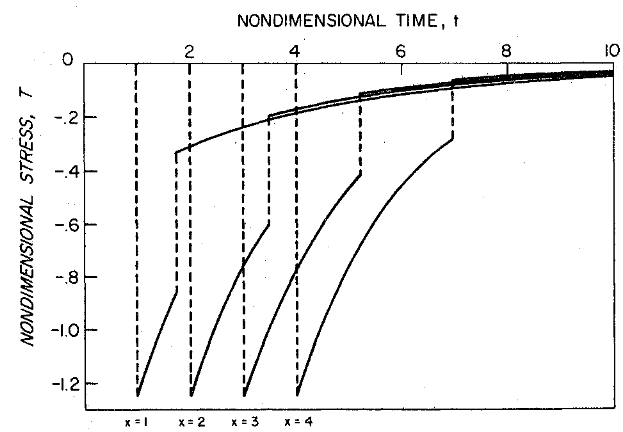

Fig. 1 Stress vs time for $q=100$, $\tau=3$, and $\epsilon=0.05$.

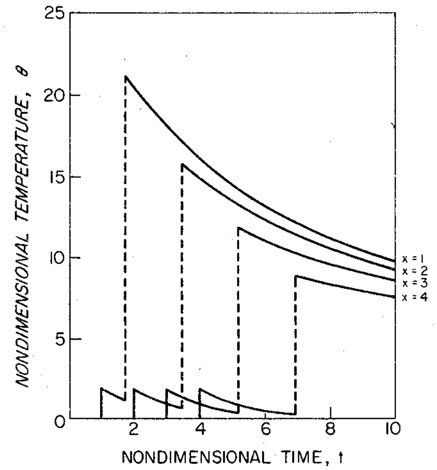

Fig. 2 Temperature vs time for $q = 100$, $\tau = 3$, and $\epsilon = 0.05$

(27). In this manner, the magnitude of the temperature across its wave-front discontinuity is given by

$$[\theta] = \frac{q}{2\sqrt{\tau}}\exp(-t/2\tau) \tag{28}$$

since $I_0(0) = 1$. This discontinuity travels with the thermal wave front and decays exponentially with time.

For the general case where $\epsilon \neq 0$, Eqs. (25) and (26) show that the temperature and the stress are discontinuous at both wave fronts. The magnitude of the stress and the temperature across the stress wave front are given by

$$[T] = \frac{\epsilon q}{2(1-\tau)} \tag{29}$$

$$[\theta] = \frac{\epsilon q \tau}{2(1-\tau)^2} \tag{30}$$

which are constants independent of time. Similarly, at the thermal wave front, the stress and the temperature suffer the following discontinuities

$$[T] = \frac{\epsilon q}{2\sqrt{\tau}(\tau-1)}\exp(-t/2\tau) \tag{31a}$$

$$[\theta] = \frac{q}{2\sqrt{\tau}}\left\{1 - \frac{\epsilon}{2(\tau-1)^2}\left[\tau(\tau+1) + \frac{(1-\tau)}{8}t^2\right]\right\}\exp(-t/2\tau) \tag{31b}$$

which also decay exponentially with time.

The solutions (25) and (26) were evaluated numerically and the results are depicted vs time in Figs. 1 and 2 for the stress and the temperature, respectively. The computations were done for $q = 100$, $\tau = 3$, and $\epsilon = 0.05$ at the four locations $x = 1$, 2, 3, and 4. In these figures, the jumps at the wave fronts are shown in broken lines.

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